If $\vec{v}, \vec{w}$ are vectors in an inner product space then:

① Cauchy-Schwarz inequality says:

$$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \, \|\vec{w}\|$$

or $$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \, \|\vec{w}\|$$

Proof:

consider the dot product

$$\langle \vec{v} + \lambda \vec{w}, \vec{v} + \lambda \vec{w} \rangle = \|\vec{v} + \lambda \vec{w}\|^2$$

for some $\lambda \in \mathbb{R}$ $\quad \geq 0$

note that
$\vec{v} \cdot \vec{v} = \|v\|^2 = \langle \vec{v}, \vec{v} \rangle$

equality when $\vec{v} \parallel \vec{w}$

$$= \langle \vec{v}, \vec{v} \rangle + 2\lambda \langle \vec{v}, \vec{w} \rangle + \lambda^2 \langle \vec{w}, \vec{w} \rangle$$

$$= \lambda^2 \langle \vec{w}, \vec{w} \rangle + 2\lambda \langle \vec{v}, \vec{w} \rangle + \langle \vec{v}, \vec{v} \rangle$$

is a polynomial in the variable λ and ≥ 0. Hence 1 root or none

[Cannot have 2 roots, it is Never Negative]

So Disc $b^2 - 4ac \leq 0$ (of quad formula)

$$[2\langle \vec{v}, \vec{w} \rangle]^2 - 4(\langle w, w \rangle)(\langle \vec{v}, \vec{v} \rangle) \leq 0$$

$$\langle \vec{v}, \vec{w} \rangle^2 - \langle \vec{w}, \vec{w} \rangle \langle \vec{v}, \vec{v} \rangle \leq 0$$

$$\langle \vec{v}, \vec{w} \rangle^2 \leq \langle \vec{w}, \vec{w} \rangle \langle \vec{v}, \vec{v} \rangle$$

$$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \, \|\vec{w}\|$$

(which was what we wanted) W^5 ✓

① AGAIN

Cauchy Schwartz proof using "Lagrange Magic"

$\|\vec{v}\| \, \|\vec{w}\| \geq |\langle \vec{v}, \vec{w} \rangle|$ says Cauchy-Schwartz inequality

proof for $\mathbb{R}^2$

Look at: $\quad \vec{v} = (x_1, y_1)$

$\|\vec{v}\|^2 \|\vec{w}\|^2$ where $\vec{w} = (x_2, y_2)$

$$= (x_1^2 + y_1^2)(x_2^2 + y_2^2)$$

$$= \underbrace{(x_1 x_2 + y_1 y_2)^2 + (x_1 y_2 - y_1 x_2)^2}$$

("multiply this out, you will see it works!) -- not obvious

$$\geq (x_1 x_2 + y_1 y_2)^2$$

$$= \langle v, w \rangle^2$$

$$\|\vec{v}\| \, \|\vec{w}\|^2 \geq \langle \vec{v}, \vec{w} \rangle^2$$

$$\rightarrow \|\vec{v}\| \, \|\vec{w}\| \geq |\langle \vec{v}, \vec{w} \rangle|$$

use this for the $\mathbb{R}^n$ proof of C.S.

Lagrange's identity:

for any 2 sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$

Then $$\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right) - \left(\sum_{k=1}^{n} x_k y_k\right)^2 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (x_i y_j - x_j y_i)^2$$

Look it up on "Wikipedia", easy to read article

(2)

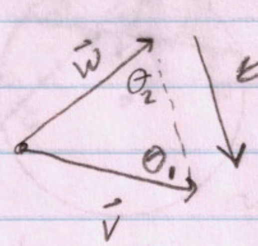
Thm

The base angles of an isosceles Δ are $=$

Proof using vectors:

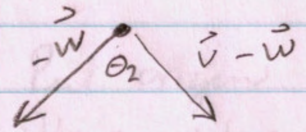
Given $\|\vec{w}\| = \|\vec{v}\|$

prove: $\theta_1 = \theta_2$



using $\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \|\vec{w}\| \cos\theta$

we can write $\dfrac{\langle -\vec{w}, \vec{v}-\vec{w} \rangle}{\|-\vec{w}\| \|\vec{v}-\vec{w}\|} = \cos\theta_2$



similarly $\dfrac{\langle -\vec{v}, \vec{w}-\vec{v} \rangle}{\|-\vec{v}\| \|\vec{w}-\vec{v}\|} = \cos\theta_1$

I will show $\cos\theta_2 = \cos\theta_1$ (which implies $\theta_2 = \theta_1$)

now $\|-\vec{w}\| = \underbrace{\|\vec{w}\| = \|\vec{v}\|}_{\text{given}} = \|-\vec{v}\|$ given

and $\|\vec{v}-\vec{w}\| = \|\vec{w}-\vec{v}\|$, so denom. values are $=$

$\langle -\vec{w}, \vec{v}-\vec{w} \rangle = \langle -\vec{v}, \vec{w}-\vec{v} \rangle$?

$\langle -\vec{w}, \vec{v} \rangle + \langle -\vec{w}, -\vec{w} \rangle = \langle -\vec{v}, \vec{w} \rangle + \langle -\vec{v}, -\vec{v} \rangle$

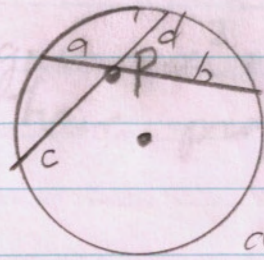
$\langle -\vec{w}, \vec{v} \rangle + \|\vec{w}\|^2 = \langle -\vec{v}, \vec{w} \rangle + \|\vec{v}\|^2$

✓ ✓

$\cos\theta_1 = \cos\theta_2 \Rightarrow \theta_1 = \theta_2$ and we have done it!

(3)

Exotic Thm:

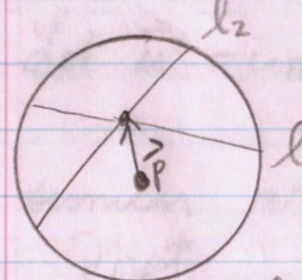


Given circle radius r and a pt P in circle, not the center - - Draw 2 chords through P and label the chord pieces a, b, c, d - Like so

claim: $ab = cd$

<u>vector proof:</u>

Put origin at center of circle.
Pts on the circle are vectors $\vec{v} \ni$
$\|\vec{v}\| = r$



$\vec{p}$ is vector from origin To point p. $\|\vec{p}\| < r$

$\vec{u}$ = unit vector in direction of l_1

$\vec{p} + t\vec{u}$ = vector equation of this line, $t \in \mathbb{R}$

What values of t make this line intersect circle?

when $\|\vec{p} + t\vec{u}\| = r$ or $\|\vec{p} + t\vec{u}\| = r^2$

$\langle \vec{p} + t\vec{u}, \vec{p} + t\vec{u} \rangle = r^2$

$\langle \vec{p}, \vec{p} \rangle + 2t\langle \vec{p}, \vec{u} \rangle + t^2\langle \vec{u}, \vec{u} \rangle - r^2 = 0$

This is a poly in variable t,

$t^2 + 2t\langle \vec{p}, \vec{u} \rangle + (\langle \vec{p}, \vec{p} \rangle - r^2) = 0$

since $\langle \vec{u}, \vec{u} \rangle = \|\vec{u}\|^2 = 1$ by defn of $\vec{u}$

next page

we have: $t^2 + 2\langle \vec{p}, \vec{u} \rangle + \langle \vec{p}, \vec{p} \rangle - r^2 = 0$

[given $x^2 + bx + c = 0$ the roots r_1, r_2
have prop that $r_1 \cdot r_2 = C$]

the poly has 2 roots, the values for t
where line hits circle. Call them
t_1, t_2

$t_1 \cdot t_2 = \langle \vec{p}, \vec{p} \rangle - r^2 < 0$ 'cause
This is independent of $\vec{u}$ $\|\vec{p}\| < r$
and $|t_1 \cdot t_2| = ab$

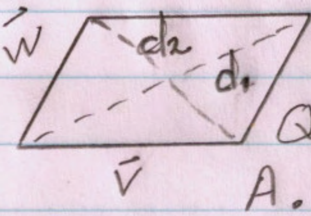
Let $\vec{u}^* =$ unit vector along l_2

similar argument will show
that
$t_1^* t_2^* = \langle \vec{p}, \vec{p} \rangle - r^2$

and $|t_1^* t_2^*| = cd$

$\rightarrow$ $ab = cd$ $\quad w^5$

(4)



Here is a parallelogram.

Q. When are the diagonals =?

A. When $\vec{w} \perp \vec{v}$

proof: Given ▱ as above Formed by $\vec{v}$, $\vec{w}$ — both not $\vec{0}$.

when is $d_1 = d_2$? Note $d_1 = \vec{v} + \vec{w}$

$d_2 = \vec{w} - \vec{v}$

or $\vec{v} - \vec{w}$, use either one

$$\|\vec{v}+\vec{w}\| = \|\vec{w}-\vec{v}\|$$

$$\|\vec{v}+\vec{w}\|^2 = \|\vec{w}-\vec{v}\|^2$$

$$\langle \vec{v}+\vec{w}, \vec{v}+\vec{w} \rangle = \langle \vec{w}-\vec{v}, \vec{w}-\vec{v} \rangle$$

$$\langle \vec{v},\vec{v} \rangle + 2\langle \vec{v},\vec{w} \rangle + \langle \vec{w},\vec{w} \rangle = \langle \vec{w},\vec{w} \rangle - 2\langle \vec{v},\vec{w} \rangle + \langle -\vec{v},-\vec{v} \rangle$$

$$= \|\vec{v}\|^2 + 2\langle \vec{v},\vec{w} \rangle + \|\vec{w}\|^2 = \|\vec{w}\|^2 - 2\langle \vec{v},\vec{w} \rangle + \|\vec{v}\|^2$$

used these facts:

$\langle \vec{u},\vec{v} \rangle = \langle \vec{v},\vec{u} \rangle$

and

$\langle -\vec{v},-\vec{v} \rangle = \langle \vec{v},\vec{v} \rangle$

$\rightarrow 4\langle \vec{v},\vec{w} \rangle = 0$

$\rightarrow \langle \vec{v},\vec{w} \rangle = 0$

$\rightarrow \vec{v} \perp \vec{w}$